Distributed SMT Solving Based on Dynamic Variable-level Partitioning

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Outline

- **Introduction**
- Dynamic Variable-level Partitioning
- Experiments & Summary

Propositional Satisfiablity (SAT)

Propositional Satisfiability (SAT): Given a propositional formula φ, test whether there is an assignment to the variables that makes φ true.

e.g., a CNF formula:

$$\varphi = (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (x_2 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$$

- The first NP-Complete problem [Cook, STOC'71]
- A core problem in computer science and a basic problem in logic

Solve a Math Problem with Arithmetic Constraints

$x > 1$

$x < 4$

$xy > 4$

$yz^2 \leq 4$

$2xz + y^2 < -20$

- Linear system
  - Simplex
  - Branch and Bound
- Non-linear system:
  - Cylindrical Algebraic Decomposition
  - **Interval Constraint Propagation**

Satisfiablity Module Theories (SMT)

Propositional Satisfiablity

$$(\neg a \lor b)$$
$$\land (c \lor d)$$
$$\land (a \lor e \lor f)$$

- Equality + UF
- Arithmetic
- Bit-vectors
- ...

$$(\neg a \lor x < -2)$$
$$\land (y > 0 \lor x^2 z + y = 3)$$
$$\land (a \lor x^2 \geq 4 \lor y > 5)$$

Boolean Skeleton of SMT Formulas

SMT:

$$(\neg a \vee x < -2) \wedge (y > 0 \vee x^2 z + y = 3) \wedge (a \vee x^2 \geq 4 \vee y > 5)$$

Boolean Skeleton:

$$(\neg a \vee b) \wedge (c \vee d) \wedge (a \vee e \vee f)$$

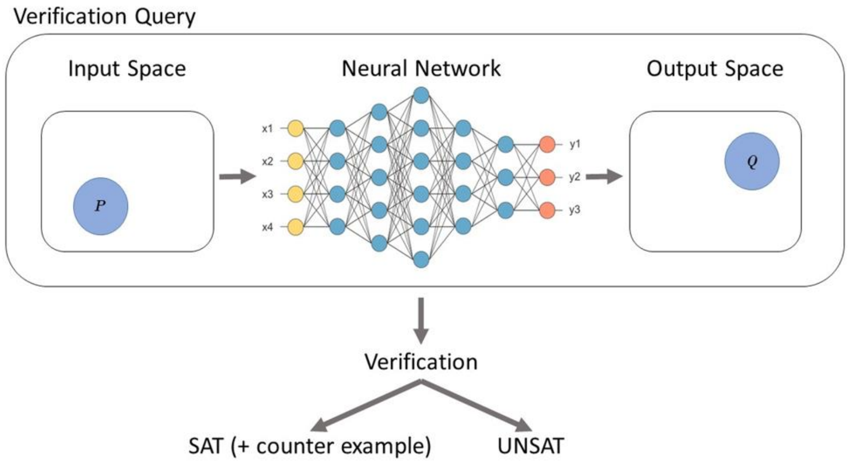
Theory Level:

$b\colon\ x < -2$ $\qquad$ $c\colon\ y > 0$

$d\colon x^2 z + y = 3$

$e\colon\ x^2 \geq 4$ $\qquad$ $f\colon\ y > 5$

Satisfiablity Module Theories (SMT)



Program Verification

Security Applications

Neural Network Verification

SMT Solvers and Solving Paradigms

SMT Solvers:

Z3 Z3++

CVC5

OpenSMT2

Yices2

…

Solving Paradigms:

- CDCL(T)
- MCSAT
- Bit-blasting
- Local Search

…

Distributed SMT Solving

- Portfolio
  - Diversification
  - Clause sharing
- **Partitioning**
  - Cube and conquer
  - Scattering

[Wintersteiger, CAV'09]

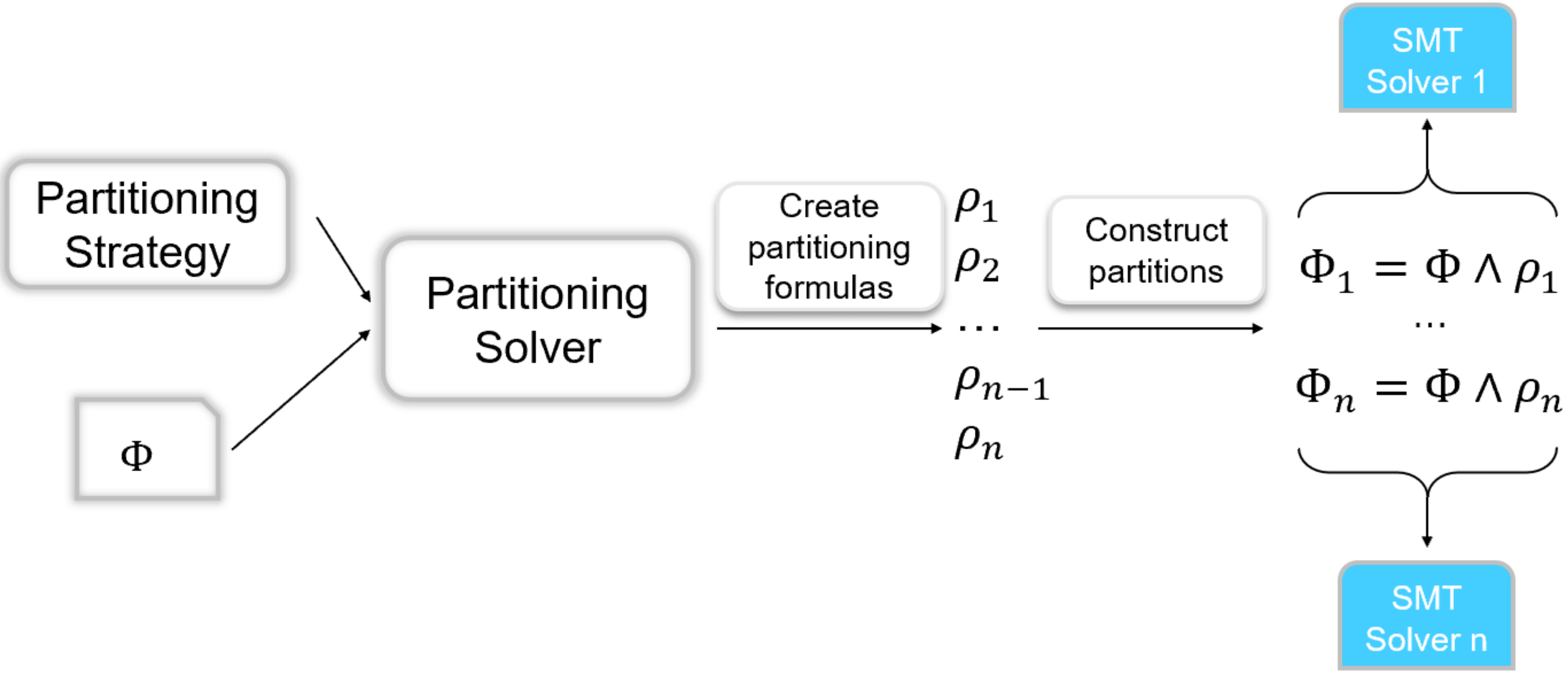
[Heule, HVC'12]

OpenSMT2 Team:
[Hyvärinen, SAT'16, FMCAD'21]
[Marescotti, LPAR'22]

CVC5 Team:
[Wilson, FMCAD'23]

Distributed SMT Solving

In partitioning strategies provided by CVC5 [Wilson, FMCAD'23], the original problem is divided into sub-problems before solving.



OpenSMT2 implements a dynamic partitioning method. [Marescotti, LPAR'22]

- partitions the instance dynamically on-demand
- shares learnt clauses

Outline

- Introduction
- **Dynamic Variable-level Partitioning**
- Experiments & Summary

Motivation of Dynamic Variable-level Partitioning

- Term-level partitioning doesn't work all the time. Pure-conjunction formulas can not be partitioned at term-level.

$$(x^2 + y^2 \leq 5) \wedge (2xy > 3) \wedge (x > -2) \wedge (y < 9)$$

- Many deep simplification technologies has not been well integrated with SMT to accelerate solving.

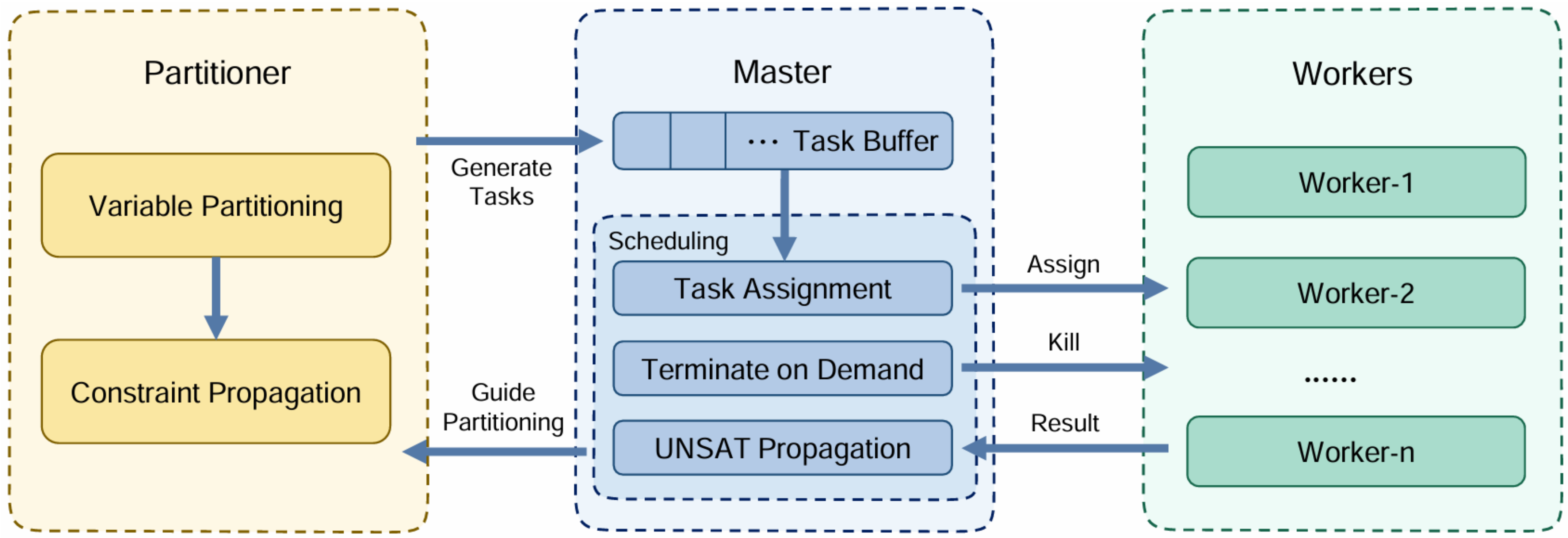
- Need a more flexible dynamic partitioning strategy.

Dynamic Distributed Framework

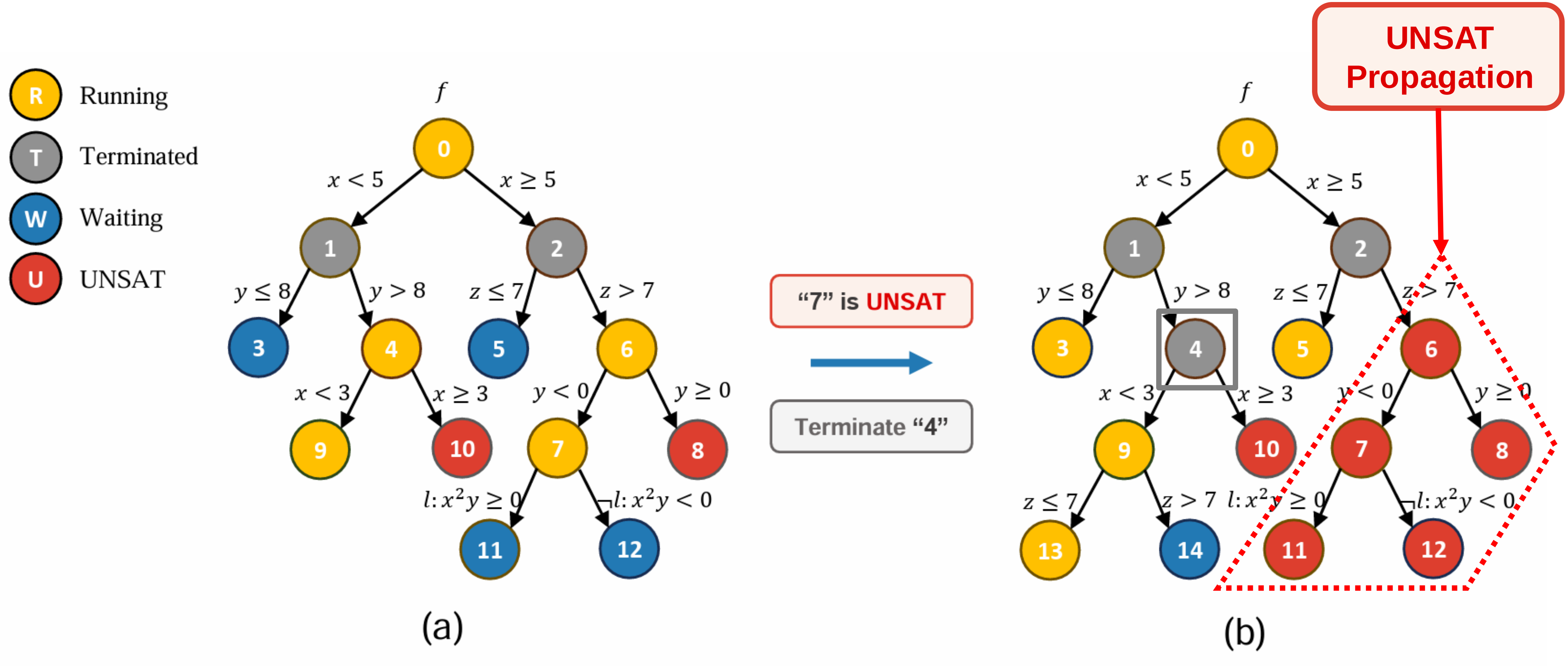
Partitioner: variable-level partitioning, sub-problems generation, and constraint propagation

Master: task assignments, on-demand terminations, and UNSAT propagations

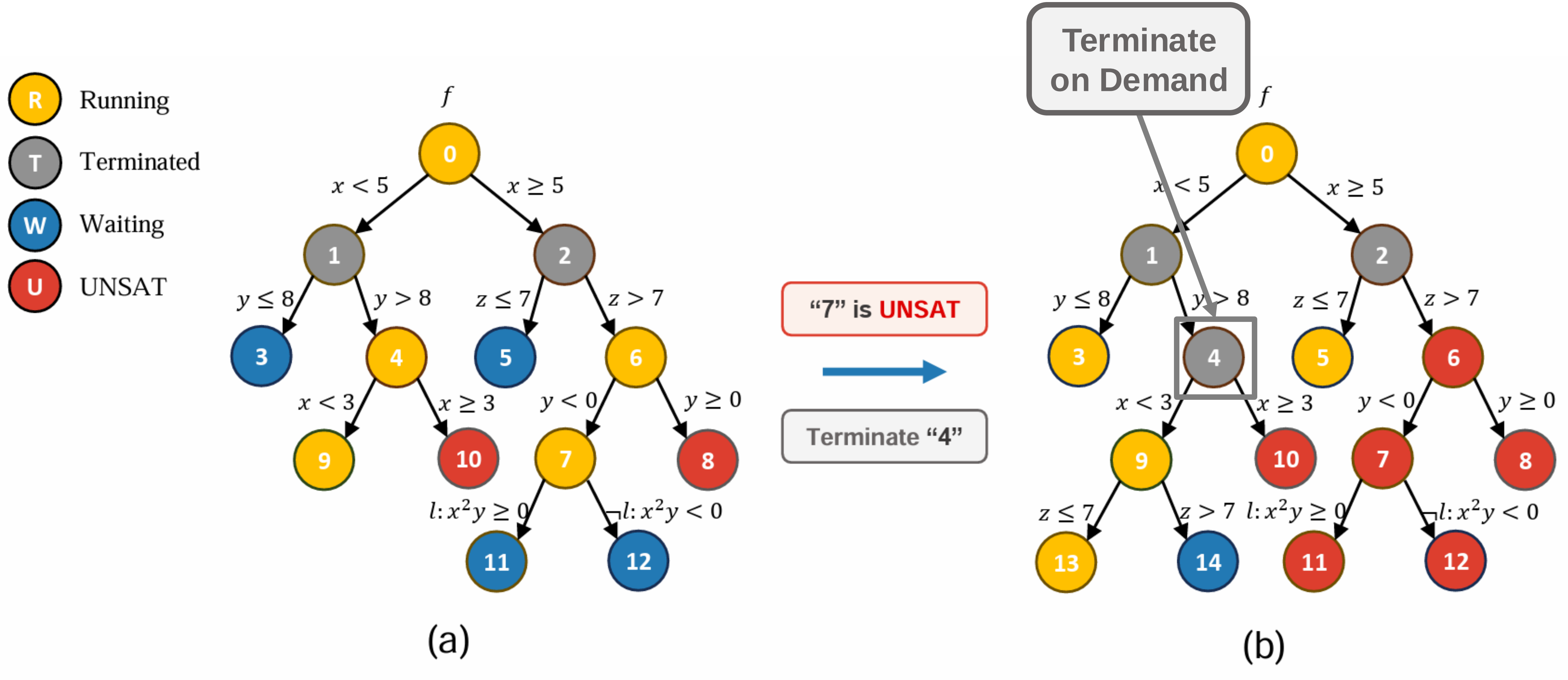
Workers: task solving and result notification



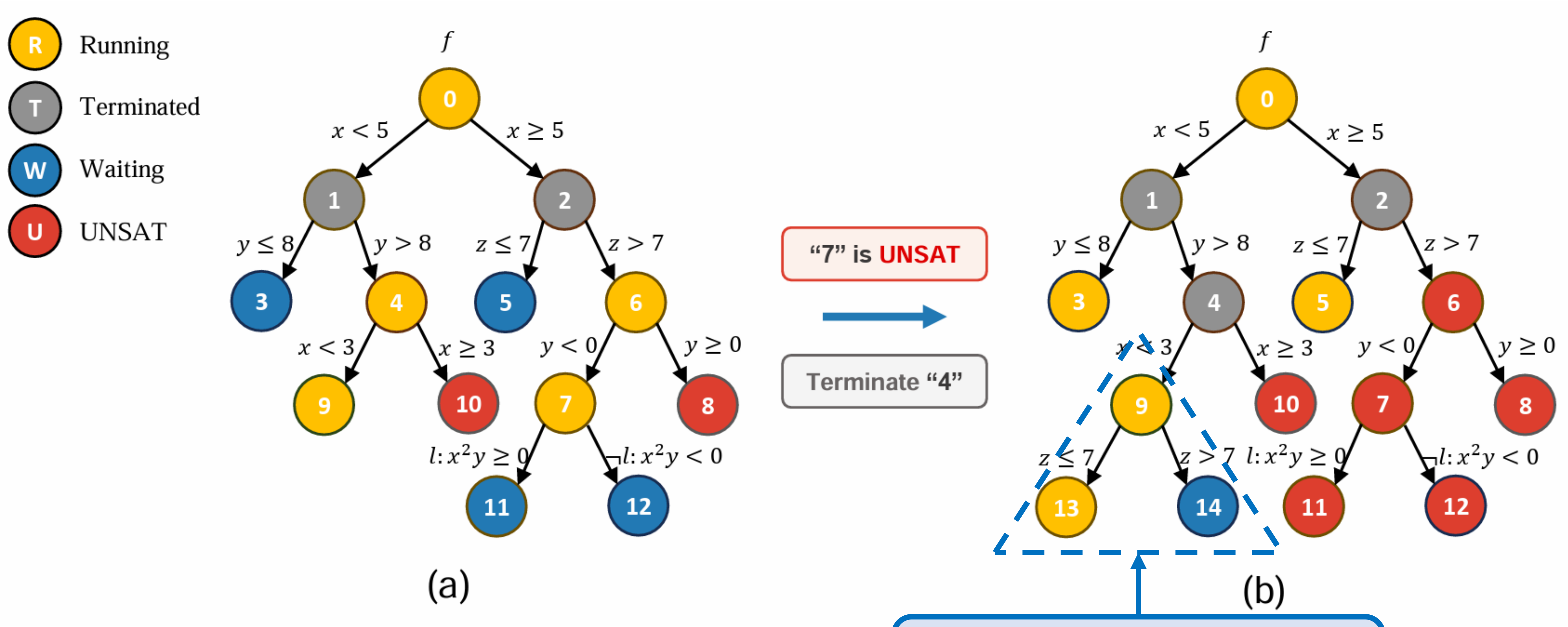
Dynamic Distributed Framework



Dynamic Distributed Framework

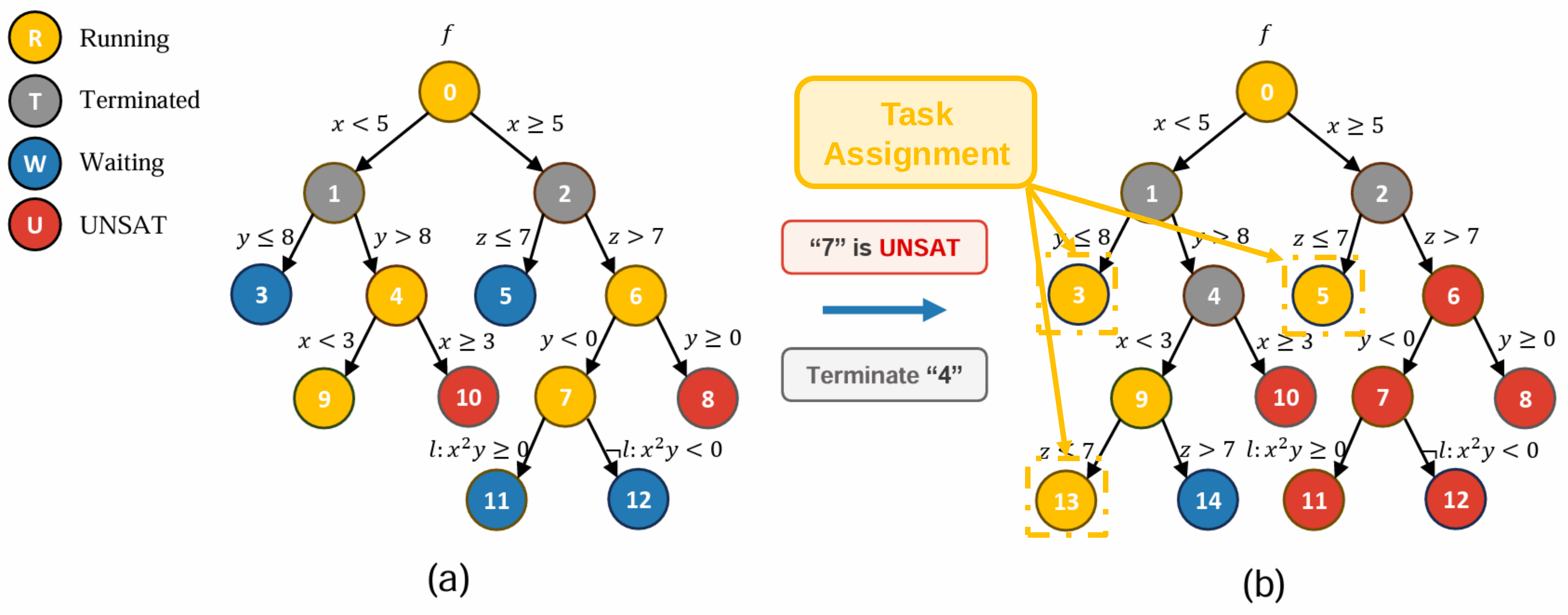


Dynamic Distributed Framework

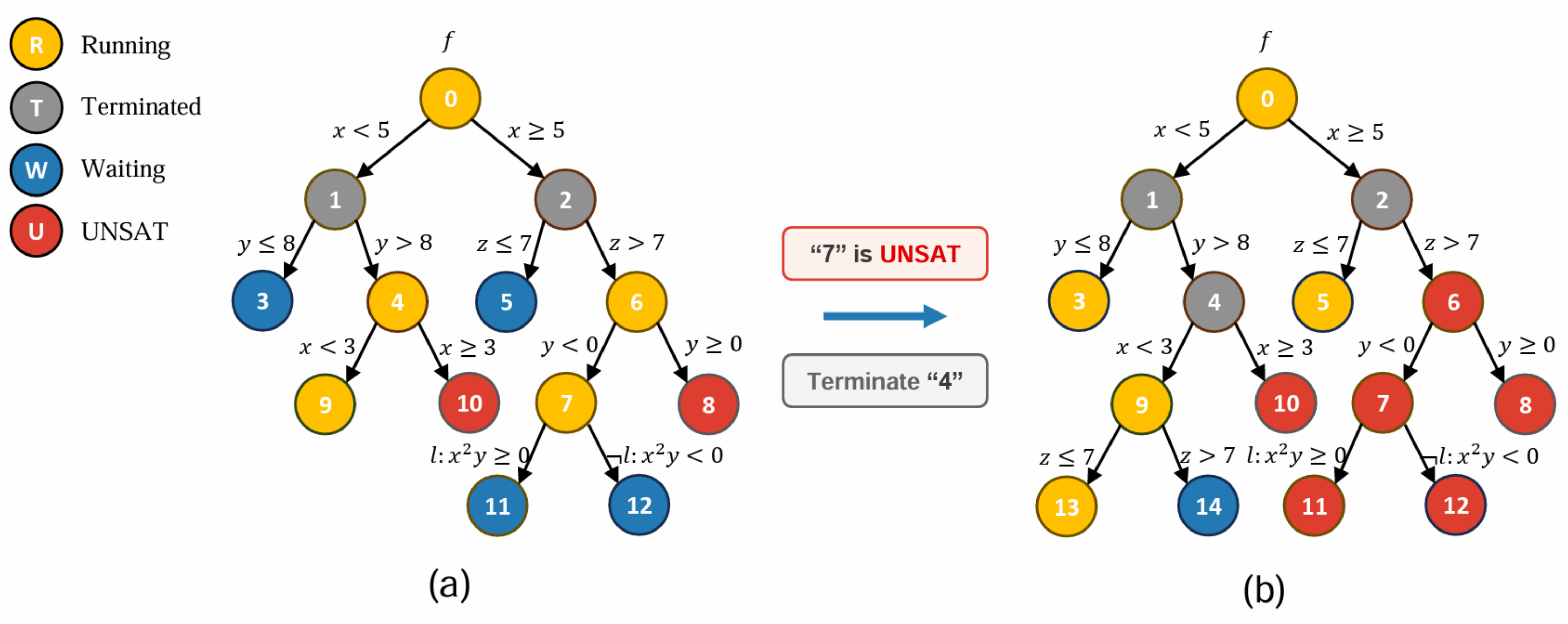


Variable-level Partitioning and Enhanced Simplification

Dynamic Distributed Framework



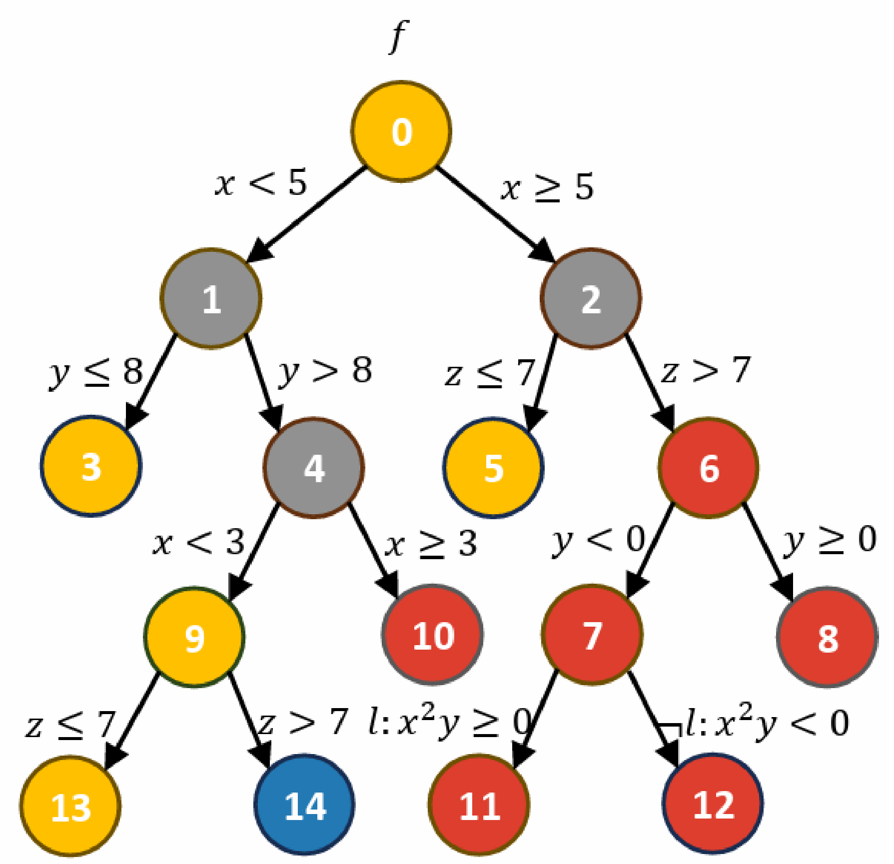
Dynamic Distributed Framework



Dynamic Variable-level Partitioning

Key Ideas

- Variable-level partitioning
- Enhanced propagation and simplification
- Dynamic distributed framework

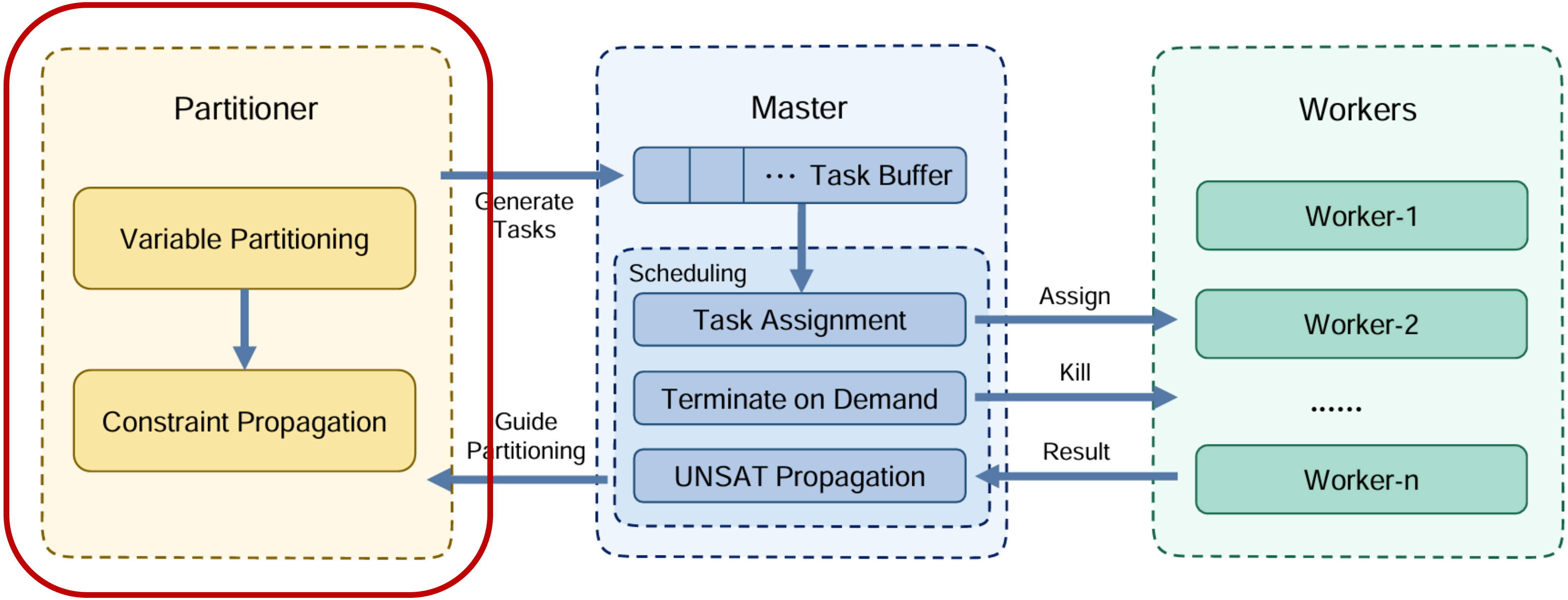


Dynamic Distributed Framework

Partitioner: variable-level partitioning, sub-problems generation, and constraint propagation

Master: task assignments, on-demand terminations, and UNSAT propagations

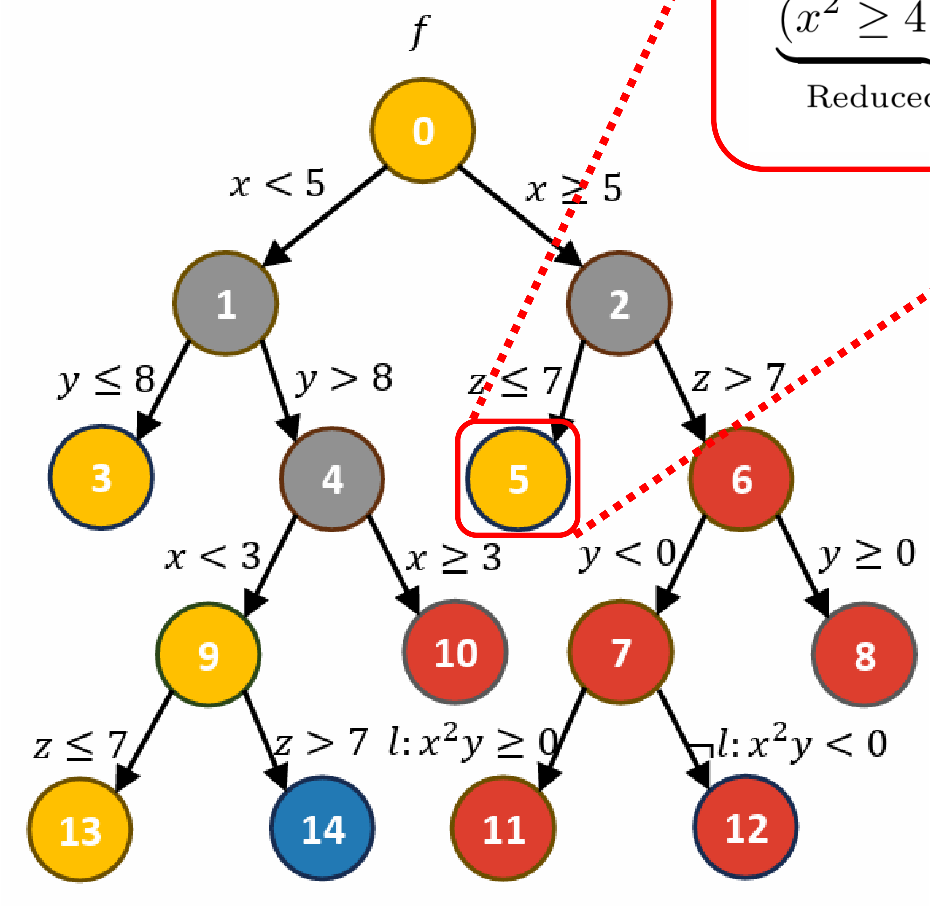
Worker: task solving and result notification



Based on the “**subpaving**” module of Z3

How does arithmetic variable-level partitioning work

Heuristic in Partition Node Selection

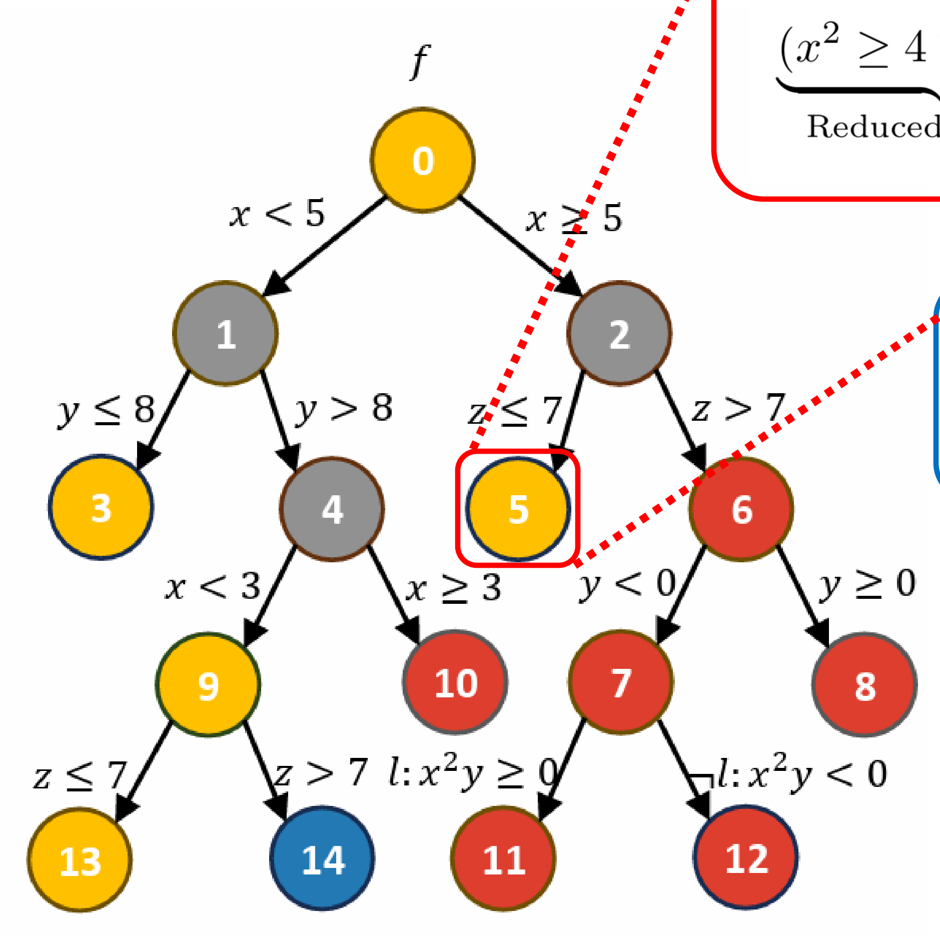


Selection of partition node 5, (Minimum tree depth, most complex formula)

$$\underbrace{(x^2 \geq 4 \vee y > 5)}_{\text{Reduced Clauses}} \wedge \underbrace{(\neg a \wedge x \in (1,4) \wedge y \in (1,\infty) \wedge z \in (-2,2))}_{\text{Feasible Domain of Variables}} \wedge \underbrace{(xy > 4 \wedge yz^2 \leq 4)}_{\text{Propagated Literals}}.$$

How does arithmetic variable-level partitioning work

Heuristic in Partition Variable Selection



Selection of partition node 5, (Minimum tree depth, most complex formula)

$$\underbrace{(x^2 \geq 4 \lor y > 5)}_{\text{Reduced Clauses}} \land \underbrace{(\neg a \land x \in (1,4) \land y \in (1,\infty) \land \boxed{z \in (-2,2)})}_{\text{Feasible Domain of Variables}} \land \underbrace{(xy > 4 \land yz^2 \leq 4)}_{\text{Propagated Literals}}.$$

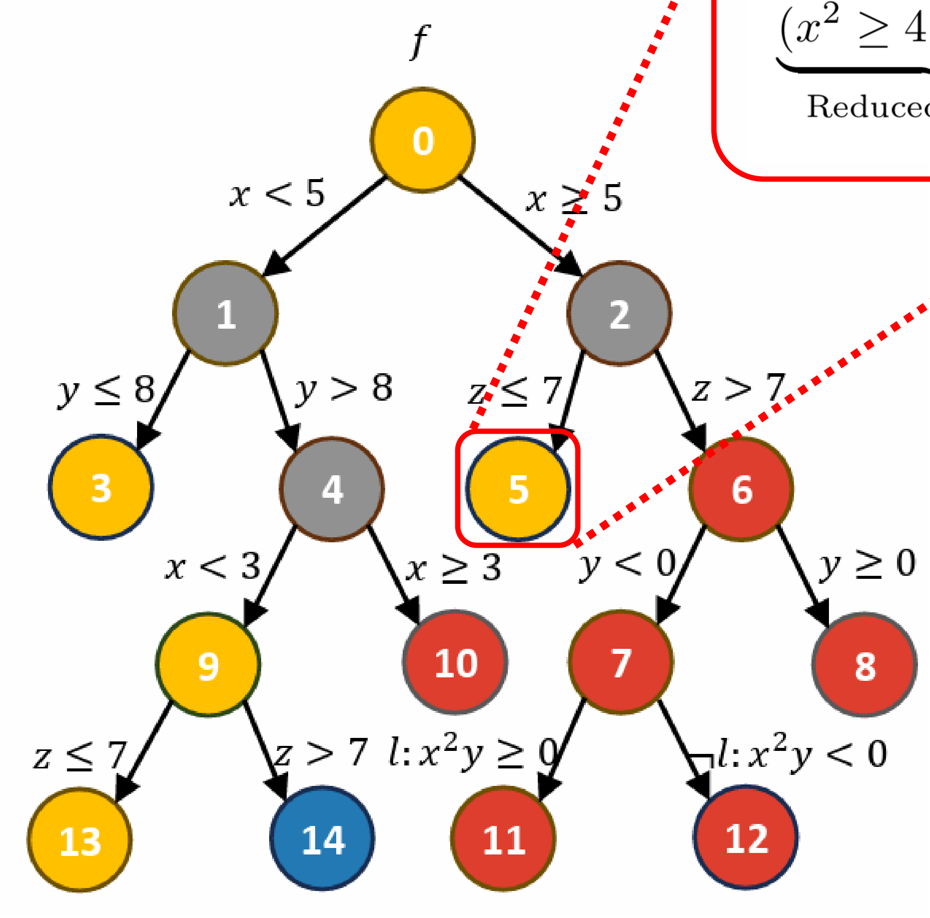
Selection of partition variable z:

$z \in (-2, 2)$

- **Highest degree**
- **Highest occurrence**
- **Least partitioning times**

How does arithmetic variable-level partitioning work

Heuristic in Partitioning at the Selected Variable



Selection of partition node 5, (Minimum tree depth, most complex formula)

$$\underbrace{(x^2 \geq 4 \lor y > 5)}_{\text{Reduced Clauses}} \land \underbrace{(\neg a \land x \in (1,4) \land y \in (1,\infty) \land \boxed{z \in (-2,2)})}_{\text{Feasible Domain of Variables}} \land \underbrace{(xy > 4 \land yz^2 \leq 4)}_{\text{Propagated Literals}}.$$

Selection of partition variable z:

$z \in \boxed{(-2, 2)}$

- **Highest degree**
- **Highest occurrence**
- **Least partitioning times**

Partitioning at the value 0:

generate two sub-tasks

$\phi_5 \land (z < 0)$ **and** $\phi_5 \land (z \geq 0)$

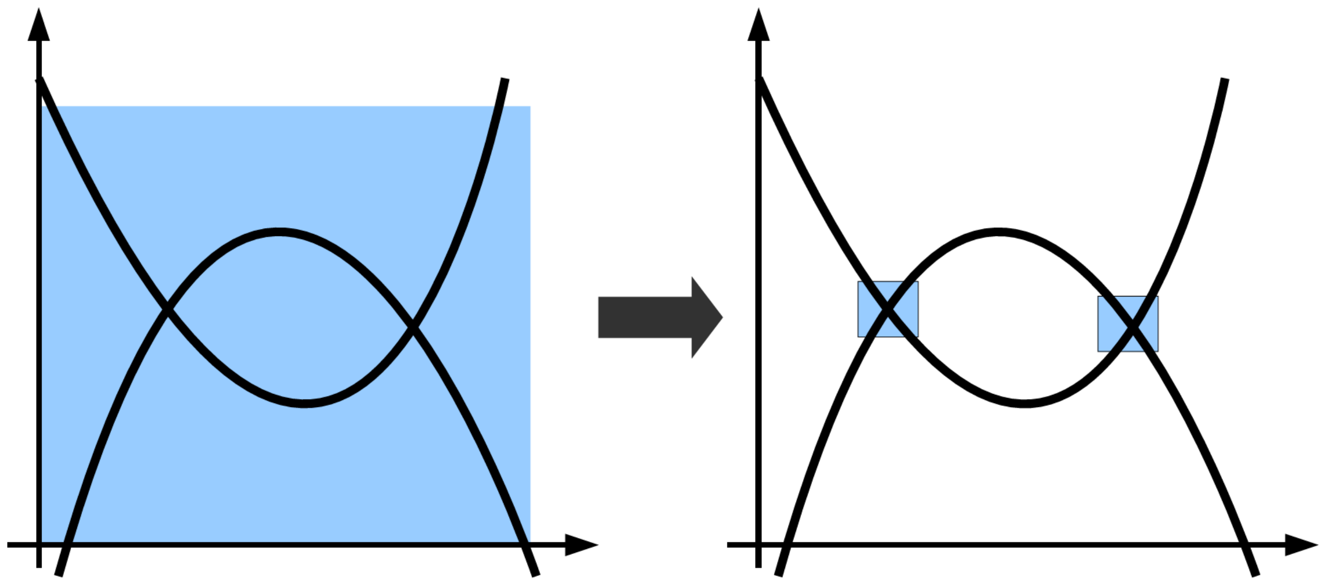
- **Value 0**
- Midpoint
- Heuristic value

Simplify Formulas via Interval Constraint Propagation (ICP)

It maintains a **feasible interval** for each variable and shrinks these intervals using simple constraint propagation.

ICP has been successfully implemented in various SMT solvers such as dReal, HySAT, and SMT-RAT.

- Shrink **variables' bounds**
- Effectively **exclude** extensive portions of **the search space**
- Sometimes proving **unsatisfiability**



[Gao, FMCAD'10]
[Schupp, Thesis'13]

Simplify Formulas via Interval Constraint Propagation (ICP)

$\Longrightarrow$ Derive $y \in (1, \infty)$ from $x \in (1, 4) \wedge xy > 4$

$\Longrightarrow$ Derive $z \in (-2, 2)$ from $y \in (1, \infty) \wedge yz^2 \leq 4$

$x > 1$

$x < 4$

$xy > 4$

$yz^2 \leq 4$

$2xz + y^2 < -20$

Simplify Formulas via Interval Constraint Propagation (ICP)

$x > 1$

$x < 4$

$xy > 4$

$yz^2 \leq 4$

$2xz + y^2 < -20$

$\Longrightarrow$ Derive $y \in (1, \infty)$ from $x \in (1, 4) \wedge xy > 4$

$\Longrightarrow$ Derive $z \in (-2, 2)$ from $y \in (1, \infty) \wedge yz^2 \leq 4$

$\Longrightarrow$ By interval arithmetic, we can obtain:

$$\begin{aligned} 2xz + y^2 &= 2 \times (1, 4) \times (-2, 2) + (1, \infty)^2 \\ &\in (-15, \infty) \end{aligned}$$

$2xz + y^2 < -20$ cannot be satisfied, when the above 4 constraints are satisfied.

Unsatisfiable!

Enhance ICP with BCP

Interval Constraint Propagation (ICP)

$x > 1$
$x < 4$
$\neg a \vee x < -2$
$a \vee xy > 4$
$a \vee y > 5$
$2xz + y^2 < -20$

We cannot simplify this formula by ICP.

Enhance ICP with BCP

Boolean Constraint Propagation (BCP) the unassigned literal in **unit clause** can only be assigned to single value to satisfy the clause.

Interval Constraint Propagation (ICP)

$x > 1$
$x < 4$
$\neg a \vee x < -2$
$a \vee xy > 4$
$a \vee y > 5$
$2xz + y^2 < -20$

We cannot simplify this formula by ICP.

Enhanced with BCP

$x > 1$
$x < 4$
$\neg a \vee x < -2$
$a \vee xy > 4$
$a \vee y > 5$
$2xz + y^2 < -20$

Simplify this formula a lot, and prove unsatisfiable directly by ICP.

Combining ICP with BCP

BICP and Formula Simplification

$x > 1$

$x < 4$

$xy > 4$

$yz^2 \leq 4$

$\neg a \vee x < -2$

$y > 0 \vee x^2 z + y = 3$

$a \vee x^2 \geq 4 \vee y > 5$

$\Longrightarrow$ Derive $x \in (1, 4),\ y \in (1, \infty),\ z \in (-2, 2)$ by ICP

$\Longrightarrow$ Infer $(x < -2) \mapsto$ False and propagate $\neg a$ by BCP

BICP and Formula Simplification

$x > 1$

$x < 4$

$xy > 4$

$yz^2 \leq 4$

$\neg a \vee x < -2$

$y > 0 \vee x^2 z + y = 3$

$a \vee x^2 \geq 4 \vee y > 5$

$\Longrightarrow$ Derive $x \in (1,4),\ y \in (1,\infty),\ z \in (-2,2)$ by ICP

$\Longrightarrow$ Infer $(x < -2) \mapsto$ False and propagate $\neg a$ by BCP

$\Longrightarrow$ Check the status of literals in the given formulas

$$
\begin{aligned}
(xy > 4 \wedge yz^2 \leq 4) &\mapsto (\text{True} \wedge \text{True}),\\
(\neg a \vee x < -2) &\mapsto (\text{True} \vee \text{False}),\\
(y > 0 \vee x^2 z + y = 3) &\mapsto (\text{True} \vee \text{Unknown}),\\
(a \vee x^2 \geq 4 \vee y > 5) &\mapsto (\text{False} \vee \text{Unknown} \vee \text{Unknown}).
\end{aligned}
$$

BICP and Formula Simplification

$x > 1$

$x < 4$

$xy > 4$

$yz^2 \leq 4$

$\neg a \vee x < -2$

$y > 0 \vee x^2 z + y = 3$

$a \vee x^2 \geq 4 \vee y > 5$

$\Longrightarrow$ Derive $x \in (1,4),\ y \in (1,\infty),\ z \in (-2,2)$ by ICP

$\Longrightarrow$ Infer (x < −2) $\mapsto$ False and propagate $\neg a$ by BCP

$\Longrightarrow$ Check the status of literals in the given formulas

$$\begin{aligned}(xy > 4 \wedge yz^2 \leq 4) &\mapsto (\text{True} \wedge \text{True}),\\ (\neg a \vee x < -2) &\mapsto (\text{True} \vee \text{False}),\\ (y > 0 \vee x^2 z + y = 3) &\mapsto (\text{True} \vee \text{Unknown}),\\ (a \vee x^2 \geq 4 \vee y > 5) &\mapsto (\text{False} \vee \text{Unknown} \vee \text{Unknown}).\end{aligned}$$

So, the formula after simplification is:

$$\underbrace{(x^2 \geq 4 \vee y > 5)}_{\text{Reduced Clauses}} \wedge \underbrace{(\neg a \wedge x \in (1,4) \wedge y \in (1,\infty) \wedge z \in (-2,2))}_{\text{Feasible Domain of Variables}} \wedge \underbrace{(xy > 4 \wedge yz^2 \leq 4)}_{\text{Propagated Literals}}.$$

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- Introduction
- Dynamic Variable-level Partitioning
- **Experiments & Summary**

Evaluation

Comparison to Sequential Solving in Arithmetic Theories Benchmarks

	QF_LRA(1753)					QF_LIA(13226)				
	SAT	UNSAT	Failed	PAR-2	Improve	SAT	UNSAT	Failed	PAR-2	Improve
CVC5(S)	958	685	110	354714	0%	7046	3212	2968	7562277	0%
CVC5(AP-p8)	980	689	84	287256	19.02%	7321	3252	2653	6791509	10.19%
CVC5(AP-p16)	980	689	84	281524	20.63%	7350	3274	2602	6678936	11.68%
CVC5(AP-p32)	982	690	81	275957	22.20%	7365	3285	2576	6603235	12.68%
OpenSMT2(S)	991	700	62	173971	0%	7985	4645	596	1994585	0%
OpenSMT2(AP-p8)	1008	701	44	132925	23.59%	8116	4660	450	1629696	18.29%
OpenSMT2(AP-p16)	1008	701	44	133043	23.53%	8138	4663	425	1555190	22.03%
OpenSMT2(AP-p32)	1009	701	43	127489	26.72%	8160	4665	401	1489780	25.31%
Z3(S)	966	680	107	316097	0%	7862	3903	1461	4025347	0%
Z3(AP-p8)	995	683	75	235645	25.45%	8055	4152	1019	3031732	24.68%
Z3(AP-p16)	996	683	74	231738	26.69%	8066	4157	1003	2983526	25.88%
Z3(AP-p32)	998	684	71	225268	28.73%	8076	4160	990	2945091	26.84%

Summary (with 8 Cores) Based on PAR-2 score		
Theory	**#Instance**	**Speed Up**
QF_LRA	1753	22.4%
QF_LIA	13226	15.7%
QF_NRA	12134	35.0%
QF_NIA	25358	32.4%

	QF_NRA(12134)					QF_NIA(25358)				
	SAT	UNSAT	Failed	PAR-2	Improve	SAT	UNSAT	Failed	PAR-2	Improve
CVC5(S)	5485	5811	838	2100561	0%	9460	4803	11095	27485835	0%
CVC5(AP-p8)	5709	5864	561	1425236	32.15%	13030	5504	6824	17250199	37.24%
CVC5(AP-p16)	5731	5864	539	1372485	34.66%	13045	5513	6800	17186305	37.47%
CVC5(AP-p32)	5743	5864	527	1343006	36.06%	13691	5588	6079	15346228	44.17%
Z3(S)	5626	5375	1133	2770153	0%	13779	5836	5743	14636656	0%
Z3(AP-p8)	5744	5686	704	1741660	37.13%	14191	6785	4382	11225626	23.30%
Z3(AP-p16)	5766	5705	663	1637352	40.89%	14193	6789	4376	11206526	23.44%
Z3(AP-p32)	5789	5712	633	1561862	43.62%	14320	6884	4154	10610746	27.51%

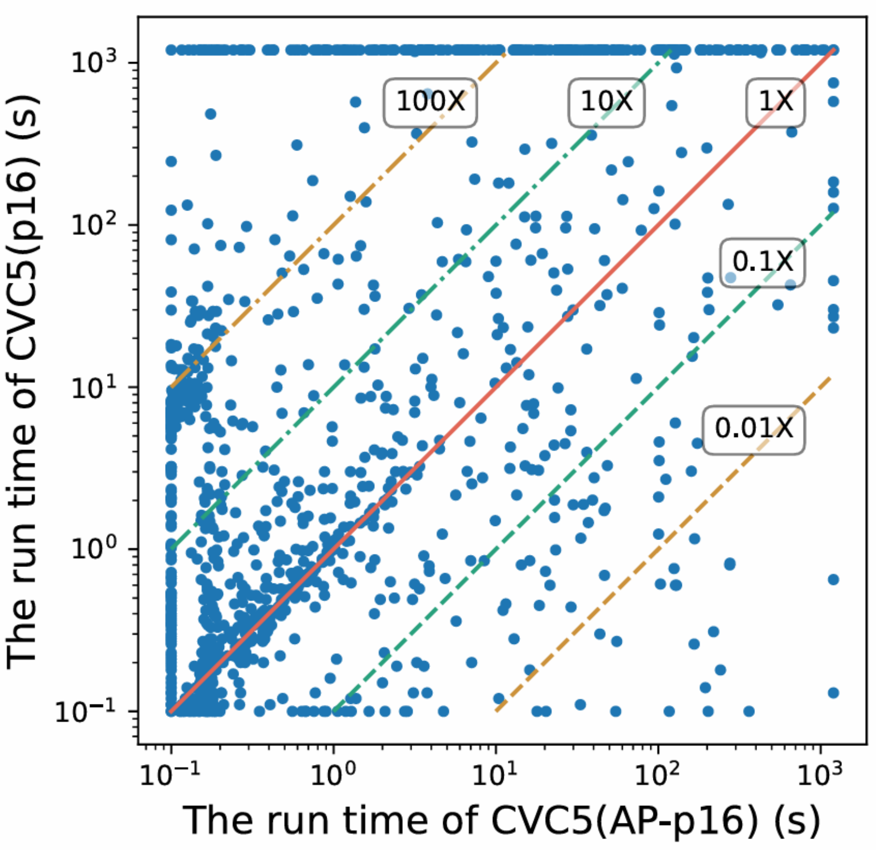
Our method with 8 cores solves **1211** additional instances (out of **6247 previously unsolved**) that **any single solver could not solve without our partitioner**.

Evaluation

Comparison to state-of-the-art partitioning strategy.

	QF_LRA(1753)				QF_LIA(13226)			
	SAT	UNSAT	Solved	PAR-2	SAT	UNSAT	Solved	PAR-2
CVC5(p8)	964	677	1641	347087	7288	3199	10487	7004216
CVC5(AP-p8)	**980**	**689**	**1669**	**287256**	**7321**	**3252**	**10573**	**6791509**
OpenSMT2(p8)	998	**701**	1699	147360	**8169**	**4698**	**12867**	**1384618**
OpenSMT2(AP-p8)	**1008**	**701**	**1709**	**132925**	8116	4660	12776	1629696
CVC5(p16)	965	676	1641	346435	7316	3225	10541	6895797
CVC5(AP-p16)	**980**	**689**	**1669**	**281524**	**7350**	**3274**	**10624**	**6678936**
OpenSMT2(p16)	997	698	1695	157708	8097	4623	12720	1778020
OpenSMT2(AP-p16)	**1008**	**701**	**1709**	**133043**	**8138**	**4663**	**12801**	**1555190**
	QF_NRA(12134)				QF_NIA(25358)			
	SAT	UNSAT	Solved	PAR-2	SAT	UNSAT	Solved	PAR-2
CVC5(p8)	5559	5798	11357	1948280	12503	4480	16983	20716252
CVC5(AP-p8)	**5709**	**5864**	**11573**	**1425236**	**13030**	**5504**	**18534**	**17250199**
CVC5(p16)	5575	5796	11371	1920929	12821	4405	17226	20123111
CVC5(AP-p16)	**5731**	**5864**	**11595**	**1372485**	**13045**	**5513**	**18558**	**17186305**

Comparison of the solving ability of the SOTA parallel strategy with CVC5 and OpenSMT2



Run time comparison in pure-conjunction instances

Summary

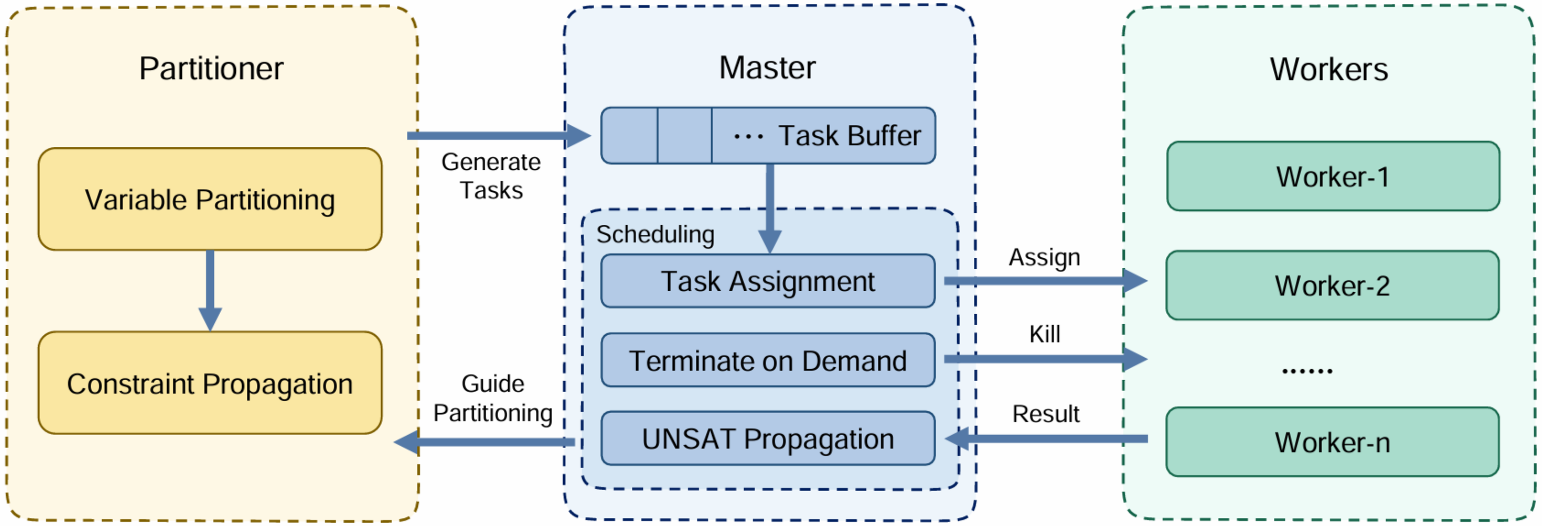
AriParti in GitHub

Parallel Version: https://github.com/shaowei-cai-group/AriParti

Distributed Version in SMT-COMP 2024: ~/Z3-Parti-Z3pp-at-SMT-COMP-2024

Key ideas:

- Variable-level partition
- BICP for simplification
- Flexible dynamic



Experimental Results:

- After being applied to the cutting-edge solvers, we **solved 3495 more instances** on average, and the solving **speed improved by about 30%**.
- Compared with the SOTA partitioning strategies, it has **significantly improved** in **nonlinear theories and pure-conjunction type instances**.

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